Chapter 4

Position Lines Theory

4.1 Introduction

In the last chapter it was established that practical procedures to enable the determination of astronomical position were required and that the theory relating to the determination of position by the technique of “Position Lines” needed to be developed for application by least squares methods. To achieve this the classical graphical approach will be reviewed and least squares based solutions considered. A least squares solution may include: the refraction effect and its rate of change; theodolite vertical collimation and its rate of change as well as latitude and longitude as unknowns to be solved for. The nature, effect and evaluation of non-random errors in time must also be considered, as must the effect on the observed vertical angle of an error in horizontal pointing.

To optimise the process observing and computing strategies need to be considered including; consideration of the observing parameters of star elevation and azimuth limits; start time of observations; the number of stars to be observed; the overall balance of stars in azimuth and altitude and the number of observations per star against number of stars.

4.2 The Theory of Position Lines

The sub-stellar point of a star (Figure 4.1) is defined as the point where the star is at the observer's zenith. In other words:

$\phi = \delta$ and

$LST = \alpha$

Where ϕ is the latitude of the point

δ is the Declination of the star

LST is Local Sidereal Time

α is the Right Ascension of the star

Therefore, the observer's longitude can be given by the formula:

$$\lambda = LST - GST$$
$$= \alpha - GST$$
$$= \alpha - UT_1 - R \pm 12^h$$

where λ is the longitude of the point
GST is Greenwich Sidereal Time
UT_1 is Universal Time corrected for earth rotation
R is the Right Ascension of the Mean Sun

In all the above, the units must consistently be those of angle or time.

If the observer observes another star, then it will be in the zenith of a sub-stellar point, point B, where:

$$\phi_B = \delta_2 \quad \text{and}$$
$$\lambda_B = \alpha_2 - UT_1 - R$$

z is the zenith distance to a star, so

$$z = 90° - h$$

where h is the vertical angle to the star.

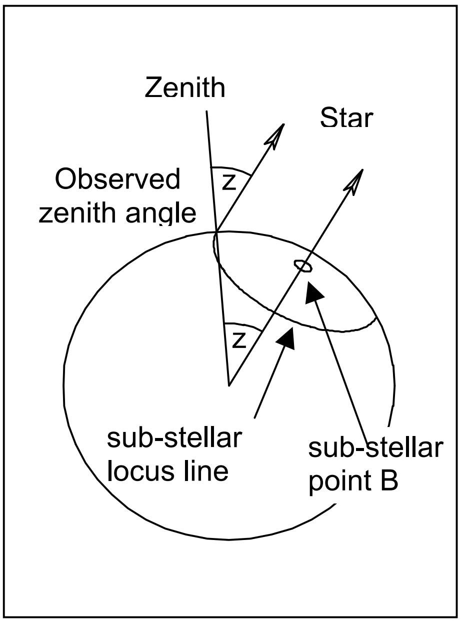

Figure 4.1 The Position Circle.

If an observer observes the star, then the observer's zenith will be on a small circle of the celestial sphere, centred on the star and with an angular radius of z. The observer's position on the earth will be somewhere on the sub-stellar locus line of that small circle (see Figure 4.1). With a second star at a different azimuth, there is a second sub-stellar locus line, crossing the first at the observer's position (see Figure 4.2).

4.2.1 Position line

In the region of the observer, the sub-stellar locus lines will be almost straight, unless z is very small. Each straight line will have a direction at right angles to the azimuth to the star. The observed zenith distance to the second star represents the spherical angular distance from the sub-stellar point of the second star to the observer's position.

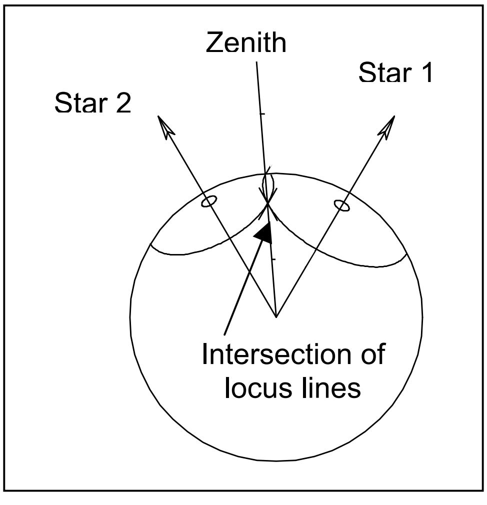

Figure 4.2 Two Position Circles.

If the observer moves towards the sub-stellar point of the second star, the observed zenith distance will decrease. If the observer moves away from the sub-stellar point of the second star, the observed zenith distance will increase.

The observer observes the zenith distance from his or her actual position. The observer can also calculate the zenith distance at any other point to the same star, at the same time. The difference between the observed zenith distance and the computed zenith distance therefore represents the angular distance (at the centre of a spherical earth) of the component of the angular separation between the actual and assumed positions, in the direction of the star.

The position solution may be derived either graphically or by least squares:

4.2.2 Computations

In the graphical method it is necessary to define a provisional value for computation of the observed position in terms of latitude and longitude ϕ_p and λ_p. The hour angle of the star at the provisional position at the observed time is given by:

$$t_p = LST_p - \alpha$$
$$= UT_1 + R + \lambda_p - \alpha$$

The computed altitude from the provisional position is given by:

$$\sin h_p = \sin \phi_p \sin \delta + \cos \phi_p \cos \delta \cos t_p$$

where

For plotting purposes, the azimuth to the star may be taken as the observed azimuth to a sufficient level of precision.

4.2.3 Plotting

The observer must be nearer the star from his provisional position, by an angular amount $(h_o - h_p)$ where h_o is the observed altitude and h_p is the altitude computed from the provisional position (see Figures 4.3 and 4.4, below).

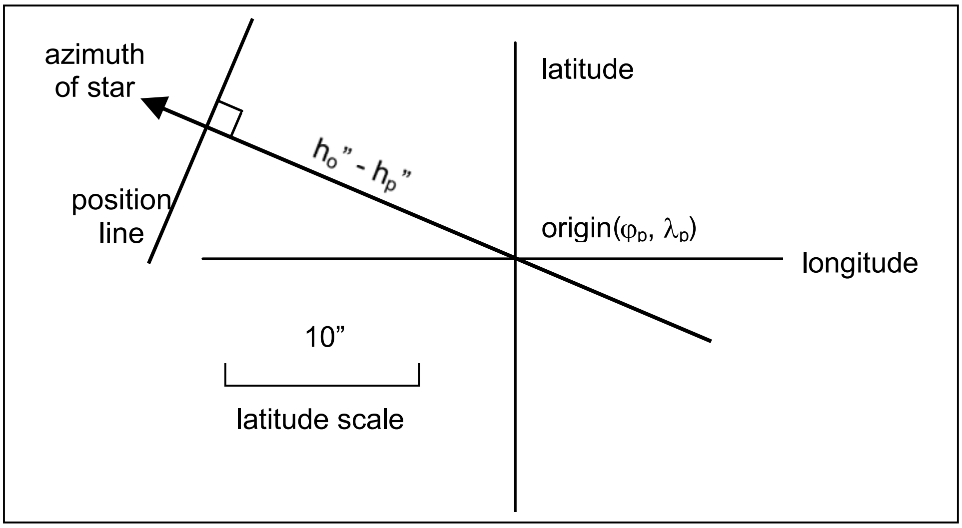

Figure 4.3   Plotting a Position Line.

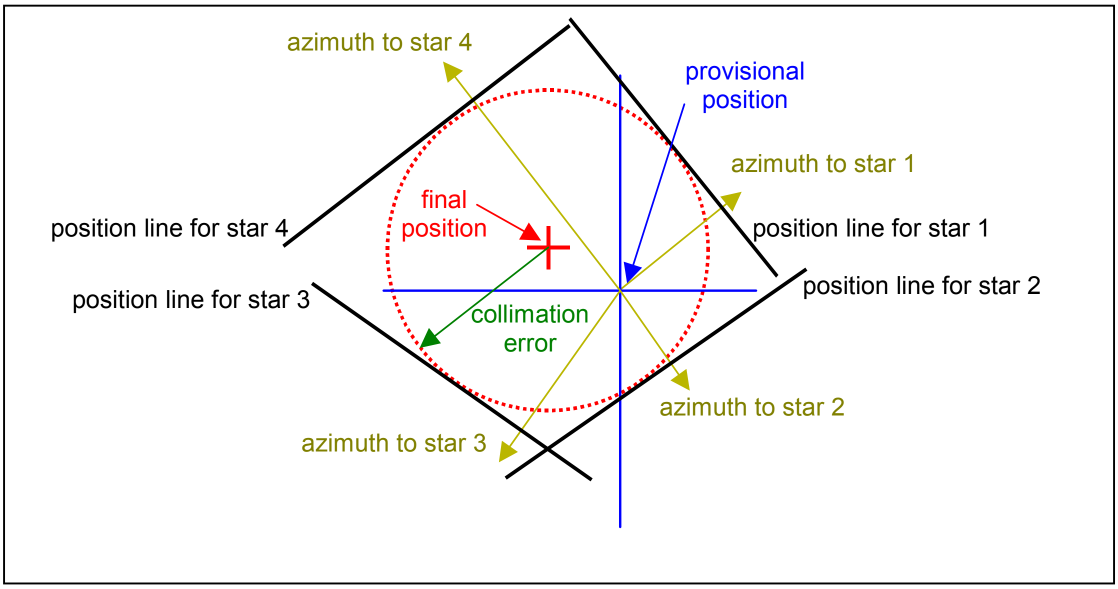

Figure 4.4   Four Position Lines surrounding the final position.

4.2.4   Accuracy

Systematic refraction and collimation errors are eliminated with four balanced stars. A systematic error in timing will give an equal error in longitude, but will give no error in latitude. With the following specification, the precision of computed position should have a standard error in the order of 3″ in latitude and 3″sec ϕ in longitude (Robbins 1976).

4.2.5 Specification

For a conventional solution, 3 sets of 6 stars may be observed. Altitudes should be greater than 40°. A one-second theodolite or better should be used. There should be 6 pointings to each star. The stars in each set should be approximately at azimuths of 45°, 135°, 225°, 315°, 90°, and 270°, each to ±10°. Altitudes should be balanced to ±5°. All observations of one set should be on the same face. The next set should be observed on the other face.

An observing programme is not necessary. In the northern hemisphere the northern star of a balanced pair should be observed first because there will usually be a star in the south, which may easily be found to balance the northern one. Because of the slower apparent movement of stars near the elevated pole, the opposite will not normally be the case.

4.2.6 Observations

The horizontal circle may be set to 0° at north by observations to Polaris. Six timed altitudes may be taken, on one face, to each star. All the readings should be made close to the centre of the horizontal cross-hair. A horizontal circle reading to the star should be made after the last vertical angle has been observed. This is for plotting and misidentification purposes. Temperature and pressure readings should be taken to enable computation of a refraction correction to the observed altitude. One set of observations may be defined as observations to 6 stars with the specification above. For the second set, the face should be changed. The stars should be observed set-by-set and not grouped by azimuth. This will ensure that all the stars of one set are more likely to be subject to the same refraction conditions.

The conventional graphical solution to position lines, described above, is suited to a solution based upon few observations. When many stars are observed, the solution can become confused if there are gross errors in the observations. Figure 4.5, below, shows a solution based on the observation of approximately 45 stars and with no gross errors in the observations. Note how the plot forms a circle around the best estimate of the solution. The circle's radius is approximately the vertical collimation of the instrument plus the average error in the refraction model. In this case, it is about 20”, which is 600 m on the ground. The least squares solution described below, with the same data, leads to an error ellipse of semi-major axis of 10 metres. It is unlikely that it would not have been possible to identify a solution by graphical means with such precision. The graphical solution, of a position computed by Position Lines, shown in Figure 4.5, below, has had all information, other than the position lines, removed.

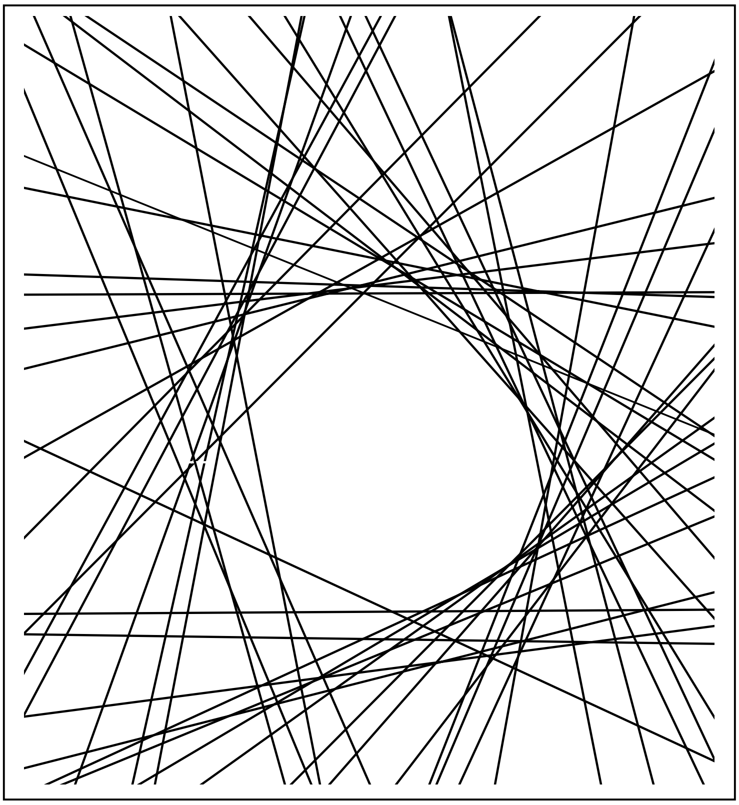

Figure 4.5 Position Line solution based on observations to 45 stars.

4.3 Least squares approach to position lines

By applying the spherical cosine formula to the astronomical triangle and taking account of the formulae in section 4.2 the observation equation may be written as:

$$\sin(h - k \cot h + c) = \sin\phi \sin\delta + \cos\phi \cos\delta \cos(UT_1 + R + \lambda - \alpha)$$

where

k is a refraction constant to be solved for and assumes that the effect of refraction is proportional to cot h

c is the vertical collimation of the instrument

The solution to the generalised least squares problem

$$\mathbf{Ax} + \mathbf{Bv} = \mathbf{b}$$

is

$$\mathbf{x} = \mathbf{(A^T(BW^{-1}B^T)^{-1}A)^{-1}A^T(BW^{-1}B^T)^{-1}b}$$

and

$$\sigma_{\mathbf{(x)}} = \mathbf{(A^T(BW^{-1}B^T)^{-1}A)^{-1}}$$

In this problem:

$$\mathbf{x} = \begin{bmatrix} \delta k \\ \delta c \\ \delta\phi \\ \delta\lambda \end{bmatrix}$$

and the units of all angular quantities are most easily expressed as radians. The equation at the beginning of this section may be rearranged as:

$$\begin{aligned} f &= \sin(h - k \cot h + c) - [\, \sin\phi \sin\delta + \cos\phi \cos\delta \cos(U + R + \lambda - \alpha) \,] \\ &= 0 \end{aligned}$$

Note the change of notation where U is the observed value of time UT_1.

$$\mathbf{A} = \begin{bmatrix} \frac{\partial f_1}{\partial k} & \frac{\partial f_1}{\partial c} & \frac{\partial f_1}{\partial \phi} & \frac{\partial f_1}{\partial \lambda} \\ \frac{\partial f_2}{\partial k} & \frac{\partial f_2}{\partial c} & \frac{\partial f_2}{\partial \phi} & \frac{\partial f_2}{\partial \lambda} \\ .. & .. & .. & .. \end{bmatrix}$$

The observations are those of vertical angle and of time, so:

$$\mathbf{B} = \begin{bmatrix} \frac{\partial f_1}{\partial h} & 0 & 0 & .. & \frac{\partial f_1}{\partial U} & 0 & 0 & .. \\ 0 & \frac{\partial f_2}{\partial h} & 0 & .. & 0 & \frac{\partial f_2}{\partial U} & 0 & .. \\ .. & .. & .. & .. & .. & .. & .. & .. \end{bmatrix}$$

which may be partitioned into two diagonal matrices:

$$\mathbf{B} = \begin{bmatrix} \mathbf{B_1} & \mathbf{B_2} \end{bmatrix}$$

The weight matrix may be divided as follows:

$$\mathbf{W} = \begin{bmatrix} \mathbf{W_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{W_2} \end{bmatrix}$$

where $\mathbf{W_1}$ refers to the observations of vertical angle (h) and

$\mathbf{W_2}$ refers to the observations of time (U)

Provided that all the observations and the corrected model are uncorrelated with each other then $\mathbf{W_1}$ and $\mathbf{W_2}$ will both be diagonal matrices.

$$\mathbf{b} = \sin(h_o - k_p \cot h_o + c_p) - [\sin\phi_p \sin\delta + \cos\phi_p \cos\delta \cos(U + R + \lambda_p - \alpha)]$$

The partial differentials are:

$$\frac{\partial f}{\partial h} = +\cos(h - k\cot h + c)\left(1 + \frac{k}{\sin^2 h}\right)$$

$$\frac{\partial f}{\partial U} = +\cos\phi\cos\delta\sin(U + R + \lambda - \alpha)$$

$$\frac{\partial f}{\partial k} = -\cot h\cos(h - k\cot h + c)$$

$$\frac{\partial f}{\partial c} = +\cos(h - k\cot h + c)$$

$$\frac{\partial f}{\partial \phi} = -\cos\phi\sin\delta + \sin\phi\cos\delta\cos(U + R + \lambda - \alpha)$$

$$\frac{\partial f}{\partial \lambda} = +\cos\phi\cos\delta\sin(U + R + \lambda - \alpha)$$

$\mathbf{B_1}$, $\mathbf{B_2}$ and $\mathbf{W}$ are diagonal matrices.

$$\mathbf{BW^{-1}B^T} = \begin{bmatrix} \mathbf{B_1} & \mathbf{B_2} \end{bmatrix} \begin{bmatrix} \mathbf{W_1^{-1}} & \mathbf{0} \\ \mathbf{0} & \mathbf{W_2^{-1}} \end{bmatrix} \begin{bmatrix} \mathbf{B_1} \\ \mathbf{B_2} \end{bmatrix}$$

$$\mathbf{BW^{-1}B^T} = \begin{bmatrix} \mathbf{B_1W_1^{-1}B_1 + B_2W_2^{-1}B_2} \end{bmatrix}$$

and so the i*th* element on the leading diagonal of $(\mathbf{BW^{-1}B^T})^{-1}$ is:

$$(B_{1ii}^2\ 206265^{-2}\ \sigma_h^2 + B_{2ii}^2\ 13751^{-2}\ \sigma_U^2)^{-1}$$

where σ_h is in seconds of arc and

σ_U is in seconds of time

and all the non-diagonal elements are 0.

The only matrix inversion is now that of $\mathbf{(A^T(BW^{-1}B^T)^{-1}A)}$ which has 4 rows and columns. Provisional values of ϕ and λ should be as good as possible (and will be in radians, given the units above). The provisional values of c and k may be taken as 0° and 0°.016

respectively. Note that in the computation of the error ellipse of the computed position, the scales of ϕ and λ are not the same. It would first be best to convert the units of σ_ϕ^2, σ_λ^2 and $\sigma_{\phi\lambda}$ all to metres before computing the error ellipse.

If observations are taken over a protracted period there is likely to be a change in meteorological conditions. A change of temperature may affect the theodolite vertical compensation mechanism. The change of refraction conditions may also affect the value of k in the observation equation. To accommodate linear changes of these parameters with time, and so avoid taking meteorological observations, the observation equation could be modified to:

$$\sin(h - (k + t\,p)\cot h + c + t\,q) = \sin\phi\sin\delta + \cos\phi\cos\delta\cos(UT_1 + R + \lambda - \alpha)$$

where p and q are coefficients and t is the time from the first observation. c and k therefore are now the refraction and collimation values at the first observation. There are now two further parameters to solve for and the **x** vector becomes:

$$\mathbf{x} = \begin{bmatrix} \delta k \\ \delta c \\ \delta p \\ \delta q \\ \delta\phi \\ \delta\lambda \end{bmatrix}$$

$$f = \sin(h - (k + t\,p)\cot h + c + t\,q) - [\sin\phi\sin\delta + \cos\phi\cos\delta\cos(U + R + \lambda - \alpha)] = 0$$

The **A** matrix becomes:

$$\mathbf{A} = \begin{bmatrix} \frac{\partial f_1}{\partial k} & \frac{\partial f_1}{\partial c} & \frac{\partial f_1}{\partial p} & \frac{\partial f_1}{\partial q} & \frac{\partial f_1}{\partial \phi} & \frac{\partial f_1}{\partial \lambda} \\ \frac{\partial f_2}{\partial k} & \frac{\partial f_2}{\partial c} & \frac{\partial f_2}{\partial p} & \frac{\partial f_2}{\partial q} & \frac{\partial f_2}{\partial \phi} & \frac{\partial f_2}{\partial \lambda} \\ .. & .. & .. & .. & .. & .. \end{bmatrix}$$

The $\mathbf{B_2}$ sub-matrix and **W** matrix are unchanged. The **b** vector becomes:

$$\mathbf{b} = \sin(h_o - (k_p + t\,p_p)\cot h_o + c_p + t\,q_p) - [\sin\phi_p\sin\delta + \cos\phi_p\cos\delta\cos(U + R + \lambda_p - \alpha)]$$

The partial differentials are:

$$\frac{\partial f}{\partial h} = + \cos(h - (k + t\,p)\cot h + c + t\,q)\left(1 + \frac{(k + t\,p)}{\sin^2 h}\right)$$

$$\frac{\partial f}{\partial U} = + \cos\phi \cos\delta \sin(U + R + \lambda - \alpha)$$

$$\frac{\partial f}{\partial k} = - \cot h \cos(h - (k + t\,p)\cot h + c + t\,q)$$

$$\frac{\partial f}{\partial c} = + \cos(h - (k + t\,p)\cot h + c + t\,q)$$

$$\frac{\partial f}{\partial p} = - t \cot h \cos(h - (k + t\,p)\cot h + c + t\,q)$$

$$\frac{\partial f}{\partial q} = + t \cos(h - (k + t\,p)\cot h + c + t\,q)$$

$$\frac{\partial f}{\partial \phi} = - \cos\phi \sin\delta + \sin\phi \cos\delta \cos(U + R + \lambda - \alpha)$$

$$\frac{\partial f}{\partial \lambda} = + \cos\phi \cos\delta \sin(U + R + \lambda - \alpha)$$

The provisional values of p and q may both be taken as 0°/hour.

The observations are those of time and vertical angle. Observations are imperfect and subject to random and systematic errors. Systematic errors due to error in the time signal and error due to the personal equation will need to be corrected for.

4.4 Non-random errors in time

Non-random errors in time include error in the time signal and errors due to the personal equation.

4.4.1 Error in the time signal

Error in the time signal will have a systematic effect on the determination of astronomic longitude at a site because all time observations will be affected by the same amount. RWM Moscow broadcasts DUT1 to $0^s.02$. On the assumption that this is correct, then UT1 can be determined to not worse than $0^s.01$, and the average error is therefore $0^s.005$. The longitude equivalent to this is 0".075, or at UK latitudes, 1.5 m. Alternatively, DUT1 can be found on the Internet in the International Earth Rotation Service's Bulletins at IERS (1998).

4.4.2 Errors due to the personal equation

The personal equation relates to the systematic error in all time observations and at all sites for an individual observer. It is caused by human interaction with the timing process. In conventional observations, the stopwatch is calibrated against a radio time signal, but real observations are of time as the stars cross the theodolite horizontal cross hair. The personal equation is therefore the difference in reaction time to the auditory stimulus of a radio time signal compared with that of the visual stimulus of a star crossing a theodolite crosshair. Such a relationship is hard to determine with any degree of certainty although experiments have been devised.

One such experiment undertaken by this author was to compare the calibration of a stopwatch using time from RWM Moscow with a calibration using time from the display of a Leica System 200 GPS receiver. If the DUT1 and dUT1 corrections are not applied to the RWM Moscow signal then both signals should give UTC. RWM broadcasts from 10 to 20 minutes past each hour and from 10 to 20 minutes to each hour. To avoid any possibility of stopwatch drift corrupting the results, the following observations were taken:

Table 4.1 Radio and GPS observations for stopwatch calibration.

Calibration with	Date 26/4/96 Period	Number of observations
RWM Moscow	$13^h\ 40^m$ - $13^h\ 50^m$	54
GPS	$13^h\ 50^m$ - $14^h\ 10^m$	108
RWM Moscow	$14^h\ 10^m$ - $14^h\ 20^m$	55

Ignoring the minutes and integer seconds, which were the same, the results were:

Table 4.2 The difference between Radio and GPS stopwatch calibrations.

	RWM Moscow	GPS	Difference
mean	$0^s.2308$	$0^s.4784$	
difference of the means			**$0^s.2476$**
standard error of the sample	$0^s.0375$	$0^s.0542$	
standard error of the mean	$0^s.0035$	$0^s.0051$	
standard error of the difference of the means			**$0^s.0062$**

There is a difference of $0^s.25$. Part of this could be explained as the difference in personal equations relating to auditory and visual stimuli but it seems excessive. From a conversation with the manufacturer's representative it is now understood that the difference is a function of the GPS processing time, which in turn is a function of the number of satellites, and will therefore vary with time. If the experiment were to be repeated using the "GPS flasher", described below, then it is likely that the results from GPS and the RWM Moscow would be more compatible.

The solution to the personal equation problem depends upon the method by which time is observed. The best solution is one where the observations do not depend upon human reaction or interpretation, that is, both time signal and star passage across the cross hair are recorded in the same medium, such as on the same videotape. Where human interaction with the observations cannot be avoided then systematic error will occur. There are several possible solutions to the evaluation of a personal equation.

4.4.2.1 Evaluation of the systematic error in the east-west component of the deviation of the vertical by astronomical and GPS observations.

Error in the computation of east-west deviation of the vertical relates to error in astronomical longitude which in turn relates to error in time. The east-west deviation of the vertical may be found from consideration of the Laplace equation:

$$A_G = A_A - (\lambda_A - \lambda_G)\sin\varphi$$

where

A_G and A_A are geodetic and astronomical azimuth respectively

λ_A and λ_G are geodetic and astronomical longitude respectively

Which, when rearranged, becomes:

$$\lambda_A = (A_A - A_G)/\sin\varphi + \lambda_G$$

If the astronomical azimuth, A_A, is found by the "azimuth by altitude" method then there are no observations of time and therefore there is no personal equation. The geodetic quantities, azimuth and longitude, are found by GPS.

The relationship between the uncertainties of the above parameters is:

$$\sigma^2_{\lambda A} = \frac{\partial f}{\partial A_A}^2\sigma^2_{AA} + \frac{\partial f}{\partial A_G}^2\sigma^2_{AG} + \frac{\partial f}{\partial \lambda_G}^2\sigma^2_{\lambda G}$$

$$= (\sin\varphi)^{-2}\sigma^2_{AA} + (\sin\varphi)^{-2}\sigma^2_{AG} + \sigma^2_{\lambda G}$$

If λ_G is found by GPS the uncertainty will be negligibly small. A_G is found from GPS and, with a relative GPS positional uncertainty of 0.005m and 1ppm of baseline length, the geodetic azimuth's uncertainty will be approximately:

$$(0.005\text{m} \pm d10^{-6})/d \text{ radians}$$

where d is the length of the line in metres.

The uncertainty of A_A will depend upon the number of observations that are taken to determine it. If the latitude is 53°, as it is in Nottingham, and A_A and A_G are each to make an equal contribution to $\sigma_{\lambda A}$, which is to be 0".5 say, then σ_{AA} and σ_{AG} must each be 0".28. This implies that the length of the GPS base line will be 14 km.

Although sufficient azimuth by altitude observations could be taken to meet a similar criterion for the astronomic azimuth, there is always the real danger of systematic horizontal refraction putting significant systematic error back into the final determination of the personal equation.

The personal equation will be the difference between the λ_A found from the position lines solution and the λ_A found from above. If both are found with uncertainty of 0".5 then the personal equation still has an uncertainty of $0^s.05$ and this is considerably greater than the uncertainty of the time signal from RWM Moscow.

4.4.2.2 Evaluation of the systematic error in the east-west component of the deviation of the vertical by precise levelling and GPS observations.

If the personal equation is ignored then a geoid model formed by astrogeodetic levelling will have a systematic east-west slope with a gradient equal to the personal equation. If points on the east and west sides of the area, for which an astrogeodetic geoid with systematic east-west slope has been determined, are connected by GPS heighting and also by precise levelling then, in principle, the systematic slope error of the geoid can be found. If the east-west distance is 20 km and if a GPS reference station is set up at the centre of the area of interest and roving receivers are at the east and west extremities of the area, then from consideration of propagation of error formulae, the ellipsoidal height difference may be found with a precision of:

$$\sqrt{2}\,(0.01\text{m} \pm 10^{-6}\,10000\text{m}) \quad = \quad 0.02 \text{ m.}$$

Likewise, 20 km of double-run precise levelling may give an orthometric height difference of:

$$\sigma_h = 0.002\sqrt{20} \text{ m}$$
$$= 0.009 \text{ m}$$

These statistics combine to give an uncertainty of a difference in the geoid model of 0.022 m. Over the 20 km east-west distance across the area, this is equivalent to an uncertainty of 0”.23 in the slope of the astrogeodetic geoid model.

4.4.2.3 Evaluation of personal equation by misclosure of an astrogeodetic levelling loop.

Imagine an astrogeodetic levelling loop that started at a point on the equator and continued eastwards until the loop closed back at the start point, i.e. the loop went around the earth at the equator. Then in the absence of error, there would be no misclosure. If there were an error in the personal equation then the misclosure would be the personal equation times the circumference of the earth. Such a method would produce a precise solution for the personal equation but the experiment is clearly impractical.

An alternative, which reduces the length of the astrogeodetic levelling loop but still involves a full 360° circumnavigation of the globe, would be to make the levelling loop follow a high parallel of latitude, say 89° 50’. Although the route would be on land and only 120 km long, the location would be difficult to get to. At the poles, each star is at a near constant altitude. Therefore, the uncertainty of the times of the vertical angles would be such that the derived east-west deviation of the vertical would be almost meaningless.

A rather more practical approach to the problem would be to examine the misclosure of an astrogeodetic levelling loop at mid-latitudes.

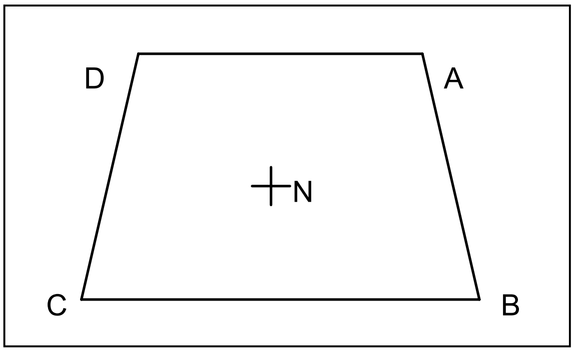

Figure 4.6 An astrogeodetic levelling loop at mid-latitudes.

In Figure 4.6, above, A and D are at the same latitude, and so are C and B. A and B are on the same meridian, and so are C and D. N is at the centre of the figure and has latitude φ_N. The latitudes of A and B are, respectively, $(\varphi_N + \delta\varphi)$ and $(\varphi_N - \delta\varphi)$. The longitude difference between A and D is λ. If R is the radius of the earth, assumed spherical for computational simplicity, with all angular terms in radians, the lengths of the sides of the figure are:

$$AB = 2\,\delta\varphi\,R$$
$$BC = \lambda\cos(\varphi_N - \delta\varphi)\,R$$
$$CD = 2\,\delta\varphi\,R$$
$$DA = \lambda\cos(\varphi_N + \delta\varphi)\,R$$

The distance, L, around the figure is the sum of the lengths of the sides:

$$L = R\,(4\,\delta\varphi + 2\,\lambda\cos\varphi_N\cos\delta\varphi)$$

which can be rearranged as

$$\lambda = (L - 4\,R\,\delta\varphi)(2\,R\cos\varphi_N\cos\delta\varphi)^{-1}$$

The distance difference west and east, over which the personal equation will have effect is:

$$d = BC - DA$$
$$= 2\,\lambda\,R\sin\varphi_N\sin\delta\varphi$$

If the misclosure is solely due to the personal equation, p, then the misclosure will be

$$m = 2\,p\,\lambda\,R\sin\varphi_N\sin\delta\varphi$$
$$= 2\,p\,(L - 4\,R\,\delta\varphi)(2\,R\cos\varphi_N\cos\delta\varphi)^{-1}R\sin\varphi_N\sin\delta\varphi$$
$$= p\,(L - 4\,R\,\delta\varphi)\tan\varphi_N\tan\delta\varphi$$

To ensure that the personal equation is found with greatest precision the figure must be optimised to maximise m for a given L. This will be achieved when:

$$\frac{\partial m}{\partial \delta\varphi} = 0$$

$$\frac{\partial m}{\partial \delta\varphi} = p\,(-\,4\ R\ \tan\varphi_N\ \tan\delta\varphi + (L - 4\ R\ \delta\varphi)\tan\varphi_N\ \cos^{-2}\delta\varphi)$$

Therefore:

$$L - 4\ R\ \delta\varphi = 4\ R\ \sin\delta\varphi\ \cos\delta\varphi$$

This equation requires an iterative solution. However, an approximate solution will suffice so, if $\delta\varphi$ is small, the above may be rewritten as

$$L - 4\ R\ \delta\varphi = 4\ R\ \delta\varphi$$

Therefore:

$$\delta\varphi = \frac{L}{8R}$$

and from substitution:

$$\lambda = L\,(4\ R\ \cos\varphi_N)^{-1}$$

$$m = \frac{p\ L^2\ \tan\varphi_N}{16\ R}$$

It is next necessary to find a value for L such that the personal equation can be significantly detected. To be at least 95% certain that p exists,

$$m > 1.96\ \sigma_m$$

If a deviation of the vertical with a standard error of 0".5 is determined every 500 m then:

$$\sigma_m = \frac{500\ 0.5\ \pi\ \sqrt{(L/500)}}{648000}$$

$$= \sqrt{L}\ 5.4\ 10^{-5}$$

Therefore from above:

$$\frac{p\ L^2\ \tan\varphi_N}{16\ R} > 1.96\ \sqrt{L}\ 5.4\ 10^{-5}$$

So, to be significant:

$$p > 1.96\ L^{-3/2}\ 5.4\ 10^{-5}\ 16\ R\ (\tan\varphi_N)^{-1}$$

$$> 10800\ L^{-3/2}\ (\tan\varphi_N)^{-1}$$

At the latitude of 53° this becomes:

$$p > 8138\ L^{-3/2}$$

Minimum significantly detectable values of p for various values of L are listed in Table 4.3, below.

Table 4.3 Lengths of astrogeodetic levelling loops required to find the magnitude of a personal equation.

p	L
$1^s.0$	232 km
$0^s.1$	1078 km
$0^s.01$	5003 km

The conclusion must be that for the value of p to be meaningfully determined and applied, the length of the astrogeodetic levelling loop would have to be of such a size that it would not fit in England.

There might possibly be an application for this method if a large number of independent routes similar to the one described in Figure 4.6 could be found. For example, 100 routes each of 1078 km could lead to the determination of p to $0^s.01$.

4.4.2.4 Evaluation of personal equation by video means

If the simultaneous sounds and sights of events are recorded on videotape then timed reaction to a series of sounds followed by timed reaction to a series of sights, and vice versa, could be used as the basis of an experiment. For example, a regular short sound, like the normal second time signal from RWM Moscow, is synchronised with a video recording of a star crossing the graticule. If there is no personal equation then the mean time of a number of consecutive audio events will be an integer number of seconds different from the mean time of the same number of consecutive visual events.

A suitable videotape might contain two minutes of audible signals and 30 simulated star crossings. Four seconds between simulated star crossings should be enough to ensure that the rhythm of visual repetition could not be used. In such an experiment, the observer times the audible signals every four seconds for the first minute and times the visual signals for the second minute. The personal equation is then the difference of the mean time of the audible events and the mean time of the visual events less the nearest integer number of seconds. In such an experiment, the observer must have the sound switched off when observing visual signals and face away from the screen when timing the audible observations.

The last solution is probably the most practical because it could be conducted in a laboratory.

4.5 The effect on the observed vertical angle of an error in horizontal pointing

In Figure 4.7, below:

O	is the observer
C	is the projection of the theodolite cross hairs onto the celestial sphere
S	star
OAB	local horizon plane
dA	angular departure of the star from the cross hairs along the horizontal hair
h	true vertical angle to the star
h_o	observed vertical angle to the star

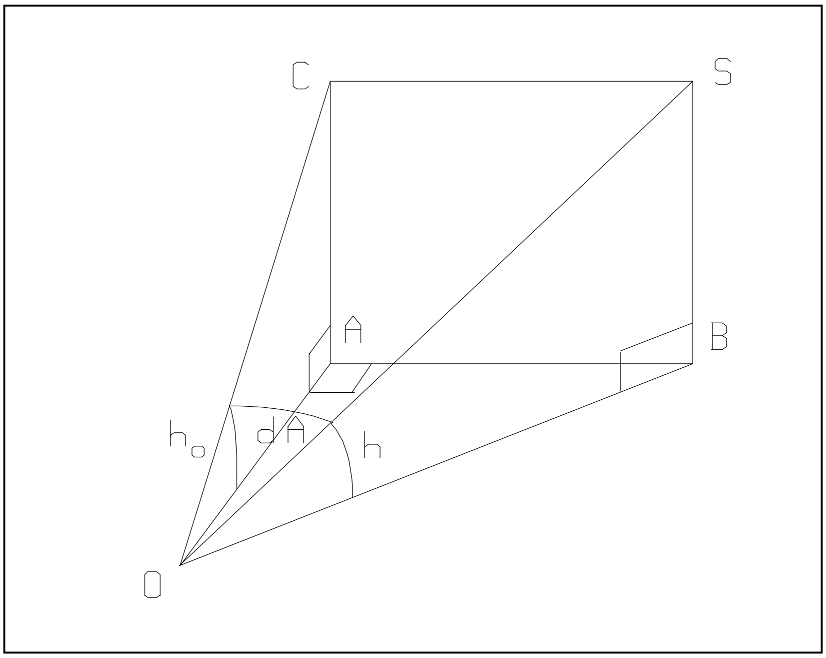

Figure 4.7 The effect on the observed vertical angle of an error in horizontal pointing.

If OB has unit length then:

$SB = \tan h$

$OS = \sec h$

$OC = \sec h \cos dA$

$AC = \sec h \cos dA \sin h_o$

Therefore since AC = SB:

$\sec h \cos dA \sin h_o = \tan h$

$\cos dA \sin h_o = \sin h$

If $h = h_o - \delta h$

where δh is the error in the observation

$\delta h = h_o - \sin^{-1}(\cos dA \sin h_o)$

dA may be taken as 2' 30", because it is a sensible minimum for the Wild T2000 theodolite. 2' 30" it is 2½ times the distance from the centre of the crosshairs to the main part of the horizontal hair. Table 4.4, below lists δh against h_o.

Table 4.4 The error in the observation of vertical angle at various vertical angles when the pointing is 2' 30" from the crosshairs.

h_o	δh
45°	0".05
60°	0".09
70°	0".15
80°	0".31
85°	0".62
88°	1".56
89°	3".12
89° 30'	6".24
89° 50'	18".47

If the altitude is greater than about 80°, it is apparent that there will be a significant systematic error in all observed vertical angles. The correction:

$$\delta h \quad = \quad h_o - \sin^{-1}(\cos dA \sin h_o)$$

will need to be subtracted from all observed vertical angles. However, if it is difficult to be consistent about the 2' 30" offset, then it will be best to avoid altitudes greater than 85°.

4.6 Investigation of a "Position Lines by Least Squares" observing and computing strategy

Before any field observations were undertaken, a computer-based study was carried out to confirm that a proposed observing and computing strategy would be workable. A prediction spreadsheet was used to create simulated observations where the observations were at the regular rate of one star every two minutes. Two minutes were chosen as that is the rate at which most observers, including the author, can observe. Changing the rate will have no significant effect upon the quality of any result, except for the amount of simulated time that is needed to achieve the result. Stars were selected so that they progressively increased in azimuth. In the field this would speed the observing process. A set of 50 stars may take about three circuits of the horizontal circle. All data sets were based upon a site near Nottingham. The maximum zenith angle was 52° with lesser values up to a zenith angle of 8°. 50 observations led to an error ellipse with a major axis of 14.5 m. The standard errors of the simulated observations were:

vertical angle 0".5 and

time $0^s.15 \cos h / (\cos \delta \cos \varphi \sin (UT1 + R + \lambda - \alpha))$

The later is a fixed value, $0^s.15$, times an expression that describes the vertical velocity of the star. The value of $0^s.15$ relates to the author's precision of reaction to a visual stimulus, a personal statistic evaluated during an investigation of alternative methods of gyrotheodolite measurements (Breach, 1985). The assumption is that a fast rising star, e.g. a star at elongation near the prime vertical, will be easier to time precisely than a star near the pole at transit which will not appear to move much in elevation.

There are two observations associated with the Position Lines method, zenith angle and time. The numerical values stated above are based upon The Wild T2000 and a hand held stopwatch. Improvement in the quality of the observations will lead to a smaller error ellipse for position, however improvement of only one observation type may have limited effect. Table 4.5 shows the relationship between the un-scaled error ellipse semi-major axis in metres, that is the figures in the shaded area, and observation quality for the particular data set. It is reasonable to suppose that a similar pattern of results would be obtained for any data set.

Table 4.5 The relationship between the un-scaled error ellipse semi-major axis in metres (figures in the shaded area), and zenith angle and time observation quality.

		Zenith angle standard error in arc seconds						
		5	2	1	0.5	0.2	0.1	0.05
Time	1	99.5	95.2	94.6	94.4	94.4	94.4	94.4
	0.5	56.7	48.3	47.6	47.3	47.2	47.2	47.2
standard	0.2	36.7	22.7	19.9	19.1	19.1	18.9	18.9
error in	0.1	32.9	15.7	11.3	9.9	9.5	9.5	9.4
arc	0.05	31.8	13.4	7.9	5.7	4.9	4.8	4.7
seconds	0.02	31.5	12.6	6.6	3.7	2.3	1.9	1.9
	0.01	31.5	12.6	6.4	3.3	1.5	1.1	1.0

Table 4.5, above, suggests that if time and zenith angle precision fall in the lighter shaded area then improvement in time quality will have more effect than improvement in zenith angle quality. If time and zenith angle precision fall in the darker shaded area then improvement in zenith angle quality will have more effect than improvement in time quality. The border between the light and dark shaded areas lies approximately where the ratio of zenith angle to time is 1 : 15, which is the numerical ratio of time and angle as circular measure i.e. $24^h : 360°$.

4.7 Options for Programmed Observations

Given a chosen site, latitude and longitude, there are a number of parameters that need to be specified to create an observing programme. They are:

Star elevation and azimuth limits
Start time
Number of stars to be observed
Overall balance of stars in azimuth and altitude
Observations per star against number of stars

A programme could be devised so that stars are observed at regular intervals. The observing process could be speeded up if the changes in altitude and azimuth between stars are minimised. A continuous clockwise rotation, i.e., successively increasing azimuth, would at least minimise horizontal circle movements. An example of part of such a programme, based upon stars of magnitude 3.5 or greater for observation with stopwatch and T2000 theodolite, is shown in tables 4.6 and 4.7, below.

Table 4.6 Parameters for a specimen observing programme.

Position line programme	
Approx latitude	51° 01’ 30”
Approx longitude	-1° 10’ 50”
Date	16 April 1996

In Table 4.7, below, the zenith angle has been corrected for standard temperature and pressure. The azimuth is such that the star will appear 2’ 30” from the central cross of the crosshairs.

Table 4.7 A specimen observing programme.

SALS	Right Ascension	Declination	Time GMT	Zenith	Az
No Mag	hh.mmss	dd.mmss	hh.mm	dd.mmss	dd.mmss
Polaris	2.2651	89.1450	0.00	39.4140	359.4730
440 2.9	16.2359	61.3120	0.02	25.0730	49.0710
494 2.4	17.5629	51.2922	0.04	39.1310	63.0640
449 3.0	16.4107	31.3637	0.06	38.2850	102.3020
442 2.8	16.2959	21.3005	0.08	44.0510	116.0220
410 2.3	15.3430	26.4339	0.10	32.1120	128.2750
416 2.7	15.4360	6.2634	0.12	50.5630	140.3820
344 2.9	12.5546	38.2053	0.14	15.3310	220.3250
316 2.2	11.4847	14.3610	0.16	43.4140	224.3530
301 2.6	11.1349	20.3314	0.18	43.1220	240.0260
280 2.6	10.1946	19.5129	0.20	51.5030	253.3350
280 2.6	10.1946	19.5129	0.22	52.0840	254.0120
300 3.1	11.0921	44.3142	0.24	28.2910	273.3150
281 3.2	10.2200	41.3138	0.26	37.4520	277.1720
257 3.3	9.2043	34.2458	0.28	51.5620	280.0400
318 2.5	11.5333	53.4331	0.30	19.4540	290.3850
245 3.1	8.5850	48.0349	0.32	47.2640	298.0920
263 3.3	9.3229	51.4210	0.34	41.1240	298.5540
297 2.4	11.0131	56.2443	0.36	27.4100	299.4510
342 1.7	12.5347	55.5923	0.38	12.2720	301.0930
352 2.4	13.2342	54.5714	0.40	8.2300	303.0010
298 1.9	11.0324	61.4651	0.42	27.4910	311.2050

The formulae associated with such a programme are stated below.

Given the following parameters:

The site in terms of latitude, and longitude

The star, in terms of its right ascension and declination

The date and time

Find: the azimuth and altitude of the star

UT1	leads to R (by computation or from tables such as SALS)	
t	=	$UT1 + R + \lambda - \alpha$
h	=	$\sin^{-1}(\cos t \cos \delta \cos \varphi + \sin \delta \sin \varphi)$
z	=	$90° - h - r_o$
A	=	$\tan^{-1}(\sin t (\sin \varphi \cos t - \cos \varphi \tan \delta)^{-1})$

where r_o is the effect of refraction.

If the altitude were to be kept constant, and therefore movement were only in azimuth, then the programme would predict when and where stars would enter or leave the small circle on the celestial sphere of given angular radius about the observer's zenith. In this case the refraction correction would be the same for all stars and could not be separated from the collimation error in the least squares solution. Therefore there could only be three unknowns to solve for at a given time and site. Observations would be irregular in time and appear randomly in azimuth.

Since stars move approximately east to west near the observer's zenith then the azimuths of the stars as they cut the small circle on the celestial sphere tend to group towards the east and the west and therefore away from the north and the south. Since error in the determined longitude is sensitive to error in time, then an east and west predominance of stars is desirable. The major decision required would be to allocate a value to the radius of the small circle. If the radius is small then the refraction and collimation factor will be small and can be well determined. However, there will only be a small number of stars that pass in and out of the small circle. In addition, the correctness of the horizontal pointing would become more important. An example of such a programme for observation with stopwatch and T2000 theodolite is in Tables 4.8 and 4.9, below.

Table 4.8 Parameters for a specimen observing programme for equal altitude observations.

Position line programme	
Approx latitude	51° 01' 30"
Approx longitude	-1° 10' 50"
Date	16 April 1996

Once again, in Table 4.9, the zenith angle has been corrected for standard temperature and pressure. The azimuth is such that the star will appear 2' 30" from the central cross of the crosshairs.

Table 4.9 A specimen observing programme for equal altitude observations.

SALS		Right Ascension	Declination	Time GMT	Zenith	Azimuth
No	Mag	hh.mmss	dd.mmss	hh.mmss	dd.mmss	dd.mmss
298	1.9	11.0324	61.4651	19.0313	19.5939	45.4111
300	3.1	11.0921	44.3142	19.2000	19.5939	97.5634
318	2.5	11.5333	53.4331	19.4636	19.5939	69.3156
257	3.3	9.2043	34.2458	20.2614	19.5939	219.5347
342	1.7	12.5347	55.5923	20.4557	19.5939	62.5405
245	3.1	8.5850	48.0349	21.0534	19.5939	273.4048
352	2.4	13.2342	54.5714	21.1555	19.5939	65.5550
344	2.9	12.5546	38.2053	21.3133	19.5939	120.5419
263	3.3	9.3229	51.4210	21.4522	19.5939	284.4402
358	1.9	13.4719	49.2026	21.4538	19.5939	82.3737
281	3.2	10.2200	41.3138	22.0831	19.5939	251.4905
300	3.1	11.0921	44.3142	23.0625	19.5939	262.1802
377	3.0	14.3151	38.1956	23.0728	19.5939	120.5821
298	1.9	11.0324	61.4651	23.1119	19.5939	314.3325
297	2.4	11.0131	56.2443	23.1644	19.5939	298.3437
318	2.5	11.5333	53.4331	24.0758	19.5939	290.4241
440	2.9	16.2359	61.3120	24.2214	19.5939	46.2850
344	2.9	12.5546	38.2053	24.2706	19.5939	239.2018
342	1.7	12.5347	55.5923	25.0845	19.5939	297.2032
352	2.4	13.2342	54.5714	25.3828	19.5939	294.1847
494	2.4	17.5629	51.2922	25.5044	19.5939	76.0843
358	1.9	13.4719	49.2026	25.5551	19.5939	277.3659
377	3.0	14.3151	38.1956	26.0250	19.5939	239.1616

The formulae associated with such a programme are below.

Given the following parameters:

The site in terms of latitude, and longitude

The star, in terms of its right ascension and declination

The date

The altitude

Find: The azimuth of the star and

East and west cross times of the small circle

$h = 90° - z - r_o$

$A = \cos^{-1}((\sin\delta - \sin\varphi \sin h)(\cos\varphi \cos h)^{-1})$

$t = \cos^{-1}((\sin h - \sin\varphi \sin\delta)(\cos\varphi \cos\delta)^{-1})$

$GST = t - \lambda + \alpha$

$UT1 = GST - R$

Problems with the computation of this programme will occur when the star path does not cross the small circle because the declination is too great or too small. The problem can be avoided if the declination range for a given altitude and latitude are first computed.

The range will be:

Maximum $\delta = 90° + \varphi - h$

Minimum $\delta = h + \varphi - 90°$

If the above computation gives a value of greater than 90° for the maximum δ in the northern hemisphere, or less than -90° in the southern hemisphere, then that implies that some stars may be at lower transit. Therefore, the numerical values of maximum and minimum δ should be taken as +90° and -90° respectively.

4.8 Polar Motion

The earth's spin axis is not fixed with respect to the earth's surface and wanders in an approximately circular motion. The movement of this axis of maximum moment of inertia cannot be predicted precisely but values of the change of the instantaneous pole with respect to the Conventional International Origin (CIO) are made available on the Internet at IERS (1998). The corrections, x and y, are in arc seconds and are, respectively, x in the direction of 0° longitude and y in the direction of 90° west longitude. Robbins (1976) quotes the corrections as:

$\Delta\phi = y \sin\lambda - x \cos\lambda$

$\Delta\lambda = -(x \sin\lambda + y \cos\lambda) \tan\phi$

The values of x and y do not normally exceed 0".3.

4.9 The effect of height above the geoid upon latitude

The downward vertical at a point curves slightly away from the equator because the equipotential surfaces are divergent towards the equator. Details are given in Robbins (1976), where a correction to the observed latitude is quoted:

$$\Delta\phi \quad = \quad -\ 0''.00017\ H \sin 2\phi$$

where H is the orthometric height

The correction is small as it amounts to only 0".17 at 45° latitude and 1000 m height.

4.10 Vertical refraction

A widely accepted model for atmospheric refraction is that of Saastamionen (1973a and 1973b). His formula is:

$$\Delta z_o'' = 16''.271 \tan z \left[1 + 0.0000394 \tan^2 z \left(\frac{p - 0.156e}{T}\right) \right] \left(\frac{p - 0.156e}{T}\right) - 0''.0000749\ p\ (\tan^3 z - \tan z)$$

Where $\Delta z_o''$ vertical angle correction
z apparent zenith distance
p total pressure (in mb)
e partial pressure of water vapour (in mb)
T temperature (°K)

With arbitrary but realistic values for p, e and T the formula could be approximated to:

$$\Delta z_o'' \quad \approx \quad k \tan z - 0''.075\ c \tan^3 z$$

where k is a function of p, e and T and is approximately 58" and c, also a function of p, e and T and is very close to 1.

The contribution of the second term can be seen in table 4.10, below.

Table 4.10 The contribution of 0".075 c $\tan^3 z$ to vertical refraction.

Zenith angle	2nd term
0°	0".0
10°	0".0004
20°	0".004
30°	0".014
35°	0".026
40°	0".044

If the theodolite is direct reading to 0".1 then there is no significant systematic error created by ignoring the second term for zenith angles less than 35°. Therefore a refraction correction model of:

$$\Delta z_o" \approx k \tan z$$

may safely be used for zenith angles of less than 35°.

4.11 Summary

In this chapter the theory of the determination of astronomical position by the technique of "Position Lines" was adapted and developed for least squares. Within the least squares solution, the effect of refraction and theodolite vertical collimation and their rates of change as well as latitude and longitude were solved for. A simple formula to model refraction has been found. Non-random errors in time were considered. The effect of an error in horizontal pointing on the observed vertical angle has been corrected for. The observing parameters of star elevation, azimuth limits, start time of observations, the number of stars to be observed, the overall balance of stars in azimuth have been considered. This influences the practical observing strategies to be discussed in Chapter 6 but any practical process will require accurate star data for the time of observation, and this is the subject of the next chapter.